

Generalized Aharonov-Bohm Effect, Homotopy Classes and Hausdorff Dimension

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June 1996

Univesité Laval preprint: LAVAL-PHY-6/96

Abstract

We suggest as gedanken experiment a generalization of the Aharonov-Bohm experiment, based on an array of solenoids. This experiment allows in principle to measure the decomposition into homotopy classes of the quantum mechanical propagator. This yields information on the geometry of the average path of propagation and allows to determine its Hausdorff dimension.

PACS index: 03.65.Bz

1. Introduction

In recent years, a number of very precise experiments have been carried out in order to test the foundations of quantum mechanics. One such fundamental property of quantum mechanics, which has never been measured experimentally, is the zig-zagness of quantum mechanical paths of propagation. Feynman and Hibbs [1] have noticed that quantum mechanical paths are non-differentiable, statistically self-similar curves. Self-similarity is closely related to scaling, which plays an important rôle in many areas of modern physics, e.g., in deep inelastic lepton-hadron scattering, Bjorken scaling in the parton model, quark distribution and splitting functions in the Altarelli-Parisi equation. Mandelbrot [2] has pointed out that self-similarity is a characteristic feature of a fractal. Fractals are characterized by a fractal dimension d_f or a Hausdorff dimension d_H . Abbot and Wise [3] have shown analytically that quantum mechanical free motion yields paths being fractal curves with $d_H = 2$. Numerical simulations [4] have shown $d_H \neq 2$ to hold for velocity dependent potentials like it occurs in Brueckner's [6] theory of nuclear matter or via dispersion relations for electrons propagating in solids [5].

Below we suggest a gedanken experiment which in principle allows to measure the Hausdorff dimension of quantum mechanical paths. In order to understand our choice of experimental set-up, let us recall how to measure the Hausdorff (fractal) dimension d_H of a fractal object in classical physics. Mandelbrot [2] considers as example the coastline of England. One takes a yardstick, representing a straight line of a given length Δx . Let ϵ denote the ratio of the yardstick length Δx to a fixed unit length l_0 . Then one walks around the coastline, and measures the length of the coast with the particular yardstick (starting a new step where the previous step leaves off). The number of steps multiplied with the yardstick length Δx gives a value $L(\epsilon)$ for the coastal length. Then one repeats the same procedure with a smaller yardstick of length $\Delta x'$, yielding the length $L(\epsilon')$. Eventually one lets Δx and hence ϵ go to zero. One observes for a wide range of length scales ϵ that the length of the British coast obeys a power law

$$L(\epsilon) = L_0 \epsilon^{-\alpha}, \quad (\epsilon \rightarrow 0). \quad (1)$$

This looks like the critical behavior of a macroscopic observable at the critical point, thus α is called a critical exponent. The Hausdorff dimension d_H is defined by

$$d_H = \alpha + 1. \quad (2)$$

As the example of the British coast line shows the determination of the Hausdorff dimension of a curve requires to measure the length of curve with many different length resolutions Δx . Then d_H is defined only in the limit $\Delta x \rightarrow 0$ via the exponent of the power law.

Now suppose we want to do a corresponding experiment to study the geometry of propagation of a massive particle in quantum mechanics. Position is an observable in quantum mechanics. Thus one can monitor a particle being emitted from a source at position x_{in} at time t_{in} to arrive at the detector at position x_{fi} at time t_{fi} by measuring its position at intermediate times t_k at regular intervals Δt . This can be done experimentally as described in Ref.[1]. An electron travelling from source to detector passes by a number of screens with holes. In order to determine by which hole the electron has passed the experimentator places a source of light behind each screen emitting photons parallel to the screen. Eventually, the photon collides with an electron having passed through a hole. From detection of the scattered photon one can determine by which hole the electron has passed. Thus one determines the length of path by joining the source to the detector by the experimentally identified holes. This gives a length $L(\Delta x)$ as a function of the resolution of length Δx , being given by the size of holes and distance between the screens. In order to extract the Hausdorff dimension one needs to send $\Delta x \rightarrow 0$, i.e., decrease the size of holes and the distance between screens by increasing the number of both.

This leads to the following problem: In order to localize with uncertainty Δx by which hole the electron has passed, one needs photon wave length $\lambda < \Delta x$. Thus by the very measurement of position the electron interacts with the photon and by collision has an uncertainty of momentum $\Delta p \geq \frac{\hbar}{\Delta x}$. When going to the limit $\Delta x \rightarrow 0$, the photon wavelength must go to zero and the uncertainty of the electron momentum (in plane of screen) Δp goes to infinity. Thus the path becomes increasingly erratic. This can be interpreted by saying that monitoring the path creates the fractal (erratic) path. Such

an experiment does not measure the geometry of propagation of a free quantum particle. This dilemma occurs for every experiment which by any means measures the position of the particle, i.e., monitors the path. One should mention also that Abbot and Wise's [3] calculation of d_H corresponds to localizing a wave packet by position measurement thus monitoring the path, i.e., not to free motion. Strictly speaking, there is no analytical calculation of d_H for unmonitored paths.

It is the central theme of this letter to propose an alternative experiment which avoids this dilemma and allows to determine the Hausdorff dimension of a free particle without monitoring the path. One can avoid to measure the position by using the concept of topology of paths. In the experiment described below one measures the interference pattern of the cross section, and deduces the contributions of homotopy classes. In each homotopy class the interaction with the vector potential of the magnetic field is analytically known. Thus one can 'reconstruct' the non-interacting case. Schulman [7] has noted the topological character of paths in the Aharonov-Bohm effect. In the Aharonov-Bohm experiment an electron is scattered from an (idealized) infinitely thin and long magnetic flux tube (solenoid). We propose a generalized Aharonov-Bohm experiment consisting of an array of such flux tubes. An array of many flux tubes is needed because the spatial resolution Δx is given in this experiment by the distance between neighbour solenoids. The determination of d_H requires $\Delta x \rightarrow 0$ hence many flux tubes need to be placed between source and detector. The array of flux tubes introduces a topology of paths. All paths (going from x_{in} to x_{fi}) can be classified by homotopy classes, given by the number and sense of winding around each of the flux tubes of the array. An electron propagating through the array of flux tubes interacts with the vector potential of the static magnetic field. However, for any path within a given homotopy class the corresponding electromagnetic interaction is a constant, which is analytically computable. The problem is to find out the relative weight of each homotopy class contributing to the propagator. This is addressed by the experiment described below.

2. Aharonov-Bohm propagator

In order to understand the proposed experiment let us review the Aharonov-Bohm experiment with a single flux tube and the corresponding calculation of the quantum mechanical propagator. For the case of the Aharonov-Bohm experiment with a single flux tube, the corresponding homotopy classes are simple and the quantum mechanical propagator in $2 - D$ (plane perpendicular to flux) can be computed analytically [8]. We consider a charged particle (charge q) passing by (scattering from) the solenoid (magnetic flux ϕ). Classically, the Lorentz force is zero. The gauge of the vector potential can be chosen such that the vector potential takes the form $A_r = 0$, $A_\theta = \phi/2\pi r$. The classical Hamiltonian in the presence of the vector potential is given by

$$H = \frac{1}{2\mu} \left(\vec{p} - \frac{q}{c} \vec{A} \right)^2, \quad (3)$$

and the action is given by

$$S = \int dt \frac{\mu}{2} \dot{\vec{x}}^2 + \frac{q}{c} \dot{\vec{x}} \cdot \vec{A}(\vec{x}, t). \quad (4)$$

Thus, when considering quantization by path integral, there occurs an Aharonov-Bohm phase factor due to the vector potential present in the action,

$$\exp \left[\frac{i}{\hbar} \int_0^T dt \frac{q}{c} \dot{\vec{x}} \cdot \vec{A} \right] = \exp \left[\frac{iq}{\hbar c} \int_{x_{in}}^{x_{fi}} d\vec{x} \cdot \vec{A} \right] = \exp[i\alpha(\theta' - \theta + 2\pi n_w)], \quad (5)$$

when the path winds $n_w = 0, \pm 1, \pm 2, \dots$ times around the solenoid, and $\alpha = q\phi/2\pi\hbar c$. This factor depends only upon the initial and final azimuthal angle θ and the number of windings, but otherwise it is independent of the path. In other words, paths can be classified by their winding number, they fall into homotopy classes. The Aharonov-Bohm propagator ($2 - D$) has been computed by Wilczek [8]. It can be decomposed into contributions corresponding to winding number n_w , given in spherical coordinates by

$$\begin{aligned} & K_{n_w}^{AB}(r', \theta'; r, \theta) \\ &= \int_{-\infty}^{+\infty} d\lambda \exp[i(\lambda + \alpha)(\theta' - \theta + 2\pi n_w)] \frac{\mu}{2\pi i \hbar T} \exp \left[\frac{i\mu}{2\hbar T} (r'^2 + r^2) \right] I_{|\lambda|} \left(\frac{\mu r r'}{i\hbar T} \right), \end{aligned} \quad (6)$$

where $I_\nu(z)$ is the modified Bessel function. The free propagator $K_{n_w}^{free}$ is given by $K_{n_w}^{AB}$ at $\alpha = 0$. Note that for each winding sector the Aharonov-Bohm propagator factorizes into

the Bohm-Aharonov phase and the free propagator,

$$K_{n_w}^{AB}(r', \theta'; r, \theta) = \exp[i\alpha(\theta' - \theta + 2\pi n_w)] K_{n_w}^{free}(r', \theta'; r, \theta). \quad (7)$$

The total propagator (sum over all windings) is

$$K^{AB}(r', \theta'; r, \theta) = \sum_{m=-\infty}^{+\infty} \exp[im(\theta' - \theta)] \frac{\mu}{2\pi i \hbar T} \exp\left[\frac{i\mu}{2\hbar T}(r'^2 + r^2)\right] I_{|m-\alpha|}\left(\frac{\mu r r'}{i\hbar T}\right). \quad (8)$$

When letting $r', r \rightarrow \infty$ the differential cross section is obtained which has been firstly computed in a different way by Aharonov and Bohm [9]. Considering $r' = r$ to be large yields velocity $v = (r' + r)/T$, momentum $p = \mu v$ and the de Broglie wave length $\lambda = 2\pi\hbar/p = \pi\hbar T/\mu r$.

The original Aharonov-Bohm effect (one solenoid) can be understood in terms of the semi-classical propagator [10]. This holds when the distance h between the solenoid and the classical path of the electron (straight line between slit and detector) is large compared to the de Broglie wave length λ , i.e., $h = \Delta x \gg \lambda$ (classical region). The semi-classical propagator is defined as the free propagator times the Aharonov-Bohm phase factor corresponding to the classical path. However, in order to determine the Hausdorff dimension, Δx needs to be sent to zero. thus the semi-classical case does not apply. We have compared numerically the Aharonov-Bohm propagator Eq.(8) with the semi-classical propagator. In Fig.[1] we show the real part of the difference as a function of α and h . We kept the following parameters fixed (given in dimensionless units): x_{in} , x_{fi} , $L = 2$ (length of straight line between x_{in} and x_{fi}), $T = 10$, $\mu = 1$, $\hbar = 1$. We have chosen the cut-off $m_{max} = 50$. From convergence tests of the free propagator, we estimate that $m_{max} = 20$ should be sufficient to guarantee stability in the sixth significant decimal digit when $h \leq 10$. The corresponding results for the imaginary part are similar. This set of parameters corresponds to the de Broglie wave length $\lambda = 10\pi$ and the crossing of the quantum mechanical region to the classical region occurs at $h = 5$. One observes that when the distance h becomes large, the difference between the Aharonov-Bohm propagator and the semi-classical propagator tends to zero. However, one observes a marked difference for small distance h ($h \rightarrow 0$).

3. Generalized Aharonov-Bohm experiment

(a) Set-up

The Aharonov-Bohm effect in the presence of one solenoid has been measured via an electron interference experiment. Here I propose a generalization: An array of N_S solenoids is positioned in a regular array with next neighbour distance Δx (see Fig.[2]). The array of solenoids is placed such that the classical trajectory coming from slit A passes through this array, while the classical trajectory coming from the slit B does not pass through this array. Like in the Aharonov-Bohm experiment one measures the interference pattern, once when all solenoids are turned off and once when all solenoids are turned on. Any change in the interference pattern is due to a change of wave function which traverses the array of solenoids. The values of ϕ_i (magnetic flux in solenoid i) are parameters to be chosen by the experimentalist (see below). The detector measures a squared modulus of the wave function $I = |\psi(\vec{x}, t)|^2$ and one observes an interference pattern.

(b) Homotopy classes

The quantum mechanical wave function can be expressed in terms of a path integral (sum over paths),

$$\psi(\vec{x}, t) = \int [d\vec{y}] \exp\left[\frac{i}{\hbar} S[\vec{y}]\right] \Big|_{\vec{x}, t; \vec{x}_0, t_0} = \sum_C \exp\left[\frac{i}{\hbar} S[C]\right], \quad (9)$$

where the sum "over histories" goes over all paths C starting from the source at \vec{x}_0, t_0 and going to the detector at \vec{x}, t passing via either one of the two slits. Because the action given by Eq.(4) has a free term and a magnetic term the wave function can be factorized

$$\psi(\vec{x}, t) = \sum_C \exp\left[\frac{i}{\hbar} S_{free}[C]\right] \exp\left[\frac{iq}{\hbar c} \int_C d\vec{y} \cdot \vec{A}(\vec{y})\right] = \sum_C K^{free}[C] \exp\left[\frac{iq}{\hbar c} \int_C d\vec{y} \cdot \vec{A}(\vec{y})\right]. \quad (10)$$

Quantum mechanical paths propagate forward in time, but can go forward and backward in space. In $D \geq 2$ dimensions paths can form loops. We have seen above that the Aharonov-Bohm propagator in a sector of fixed winding number is given by the free propagator ($\alpha = 0$) in this winding sector times the Aharonov-Bohm phase factor. This Aharonov-Bohm phase

factor is the same for all those paths which can be mapped onto each other by stretching and deformation without crossing the solenoid. The winding number n_w is a topological quantum number which characterizes the paths. The full Aharonov-Bohm propagator is given by the sum over all winding sectors. All this carries over to the generalized Aharonov-Bohm experiment with an array of N_S solenoids. The full propagator decomposes into homotopy classes. In each homotopy class the propagator factorizes into the free propagator in this homotopy class and a generalized Aharonov-Bohm phase factor, given in analogy to Eq.(5) by

$$\exp \left[\frac{iq}{2\pi\hbar c} [(\theta' - \theta)\phi_{tot} + 2\pi[n_1\phi_1 + \cdots n_{N_S}\phi_{N_S}]] \right], \quad \phi_{tot} = \phi_1 + \cdots \phi_{N_S}. \quad (11)$$

The topologically different (homotopy) classes are characterized by the winding numbers n_1, \cdots, n_{N_S} , with $n_i = 0, \pm 1, \pm 2, \cdots$. Because Maxwell's theory is an Abelian gauge theory, homotopy classes do not depend on the sequential order of winding around individual solenoids. Equivalent paths with the same winding, but different sequential order are shown in Fig.[3].

The decomposition property of the propagator for a fixed homotopy class has the following implication being important for the experiment: Changing the magnetic flux in the solenoid $\phi \rightarrow \phi'$ and hence $\alpha \rightarrow \alpha'$, changes the Aharonov-Bohm phase factor in each homotopy class and hence the total Aharonov-Bohm propagator. But it *does not change* the free propagator in each homotopy class. Thus experimentally, one has a handle to measure the free propagator corresponding to a given homotopy class. We introduce a cut-off in the winding numbers $n_i < n_{cut-off}$. This is based on the assumption that winding numbers beyond the cut-off give contributions to the amplitude which are in the order of experimental errors and hence can not be detected. This cut-off makes the number of homotopy classes finite. Let us enumerate the homotopy classes by $h = 1, 2, \cdots, N_H$. The experimentalist chooses a set of fluxes of the solenoids: $\phi_i^{(1)}, i = 1, \cdots, N_S$ and measures the corresponding interference pattern, say $I^{(1)}$. Then the experimentalist chooses another set of fluxes of the solenoids, $\phi_i^{(2)}, i = 1, \cdots, N_S$, and measures again the interference pattern, $I^{(2)}$. This is repeated for N_F different sets of fluxes. The information obtained is then sufficient to

determine the free propagators in the homotopy classes $h = 1, \dots, N_H$. Substituting the phase factor, Eq.(11), into the wave function, Eq.(10), yields the intensity for N_F different sets of fluxes,

$$I^{(f)} = \left| \sum_h K_h^{free} \exp \left[\frac{iq}{2\pi\hbar c} [(\theta' - \theta)\phi_{tot} + 2\pi(n_1\phi_1^{(f)} + \dots n_{N_S}\phi_{N_S}^{(f)})] \right] \right|^2, \quad f = 1, \dots, N_F \quad (12)$$

Because a given set of fluxes and a given homotopy class h determines the generalized Aharonov-Bohm phase factor, and the free propagator in each homotopy class K_h^{free} is independent from the fluxes, this equation allows to determine the unknown coefficients K_h^{free} for $h = 1, \dots, N_H$. Because K_h^{free} are complex numbers, and vector potentials \vec{A} and fluxes ϕ and are real, we need at least twice as many sets of fluxes as the number N_H of homotopy classes considered, $N_F > 2N_H$.

(c) Length of paths and Hausdorff dimension

Suppose we have performed the above experiment and we know the free propagator K_h^{free} for homotopy classes $h = 1, \dots, N_H$. From that we can construct the length of an average quantum mechanical path in the following way. Classically, one defines a length of a particle moving along a trajectory (from $\vec{x}_{in} = \vec{x}(t_{in})$ to $\vec{x}_{fi} = \vec{x}(t_{fi})$) by

$$L[x(t_1), x(t_2), \dots, x(t_{N-1})] = \sum_{k=0}^{N-1} | \vec{x}(t_{k+1}) - \vec{x}(t_k) | \quad (13)$$

and takes the limit $\Delta t \rightarrow 0$ in the end. In quantum mechanics, position and length are observables. In analogy to the classical mechanics the definition of length of trajectories in quantum mechanics also involves the position (observable) at different times. In quantum mechanics this requires to consider a transition amplitude from some initial state $|\psi_{in}\rangle$ at $t = t_{in}$ to some final state $|\psi_{fi}\rangle$ at $t = t_{fi}$. According to Feynman and Hibbs [11] the transition element for any function $F[x(t_1), x(t_2), \dots, x(t_{N-1})]$ of position x at different time steps t_1, \dots, t_{N-1} is given by

$$\langle \hat{F} \rangle = \frac{\langle \psi_{fi}(t_{fi}) | \hat{F}[x(t_1), \dots, x(t_{N-1})] | \psi_{in}(t_{in}) \rangle}{\langle \psi_{fi}(t_{fi}) | \psi_{in}(t_{in}) \rangle}$$

$$= \frac{\int [Dx(t)] dx_{fi} dx_{in} \psi_{fi}^*(x_{fi}) F[x(t_1), \dots, x(t_{N-1})] \exp[\frac{i}{\hbar} S] \psi_{in}(x_{in})}{\int [Dx(t)] dx_{fi} dx_{in} \psi_{fi}^*(x_{fi}) \exp[\frac{i}{\hbar} S] \psi_{in}(x_{in})}. \quad (14)$$

Feynman and Hibbs call this a weighted average. It can be interpreted as a sum over all paths of the observable F multiplied with the weight of the exponential action. Note that although this has an interpretation as path integral the matrix element is a quantum mechanical expression which can be defined via the Schrödinger equation. Substituting F by the classical length, Eq.(13), and choosing position eigenstates as initial and final states, one obtains ($x_k \equiv x(t_k)$)

$$\begin{aligned} \langle \hat{L}(\Delta t) \rangle &= \langle \sum_{k=0}^{N-1} |x_{k+1} - x_k| \rangle \\ &= \frac{\int dx_1 \cdots dx_{N-1} \sum_{k=0}^{N-1} |x_{k+1} - x_k| \exp[\frac{i}{\hbar} S[x_k]]}{\int dx_1 \cdots dx_{N-1} \exp[\frac{i}{\hbar} S[x_k]]} \\ &= \frac{\sum_C L_C \exp[\frac{i}{\hbar} S[C]]}{\sum_C \exp[\frac{i}{\hbar} S[C]]}. \end{aligned} \quad (15)$$

The last equation is a short hand notation. Note that each curve C corresponds to pieces of straight line joining positions at adjacent times, i.e., $x(t_{k+1})$ with $x(t_k)$ for $k = 0, 1, \dots, N$. Note, however, that this expression is not well defined in the limit $\Delta t \rightarrow 0$. The average path is a fractal, hence its length becomes infinite! This is an example, where an infinity occurs in the continuum limit of non-relativistic quantum mechanics. We need to introduce a regularization. A natural regularization of the transition element expressed via the path integral is that given in Eq.(15) where Δt is kept finite. However, such regularization is not suitable for the proposed experiment because there is *no* measument taken at regular time intervals Δt . On the other hand the experimentator has at his disposal the spatial resolution Δx , i.e., the distance between neighbour solenoids. The resolution Δx comes from an array of flux tubes. We have seen in the previous sections that the path integral can be decomposed into corresponding homotopy classes, counting the orientation and winding number around each solenoid. Thus in analogy to the regularization of the path integral via finite temporal resolution Δt by Eq.(15), we define a regularization via finite spatial resolution Δx by

$$\langle \hat{L}(\Delta x) \rangle = \frac{\sum_{h=1}^{N_H} L(h) \exp[\frac{i}{\hbar} S[h]]}{\sum_{h=1}^{N_H} \exp[\frac{i}{\hbar} S[h]]}, \quad (16)$$

where $h = 1, \dots, N_H$ denotes the homotopy classes, N_H is the cut-off determined from experiment, $\exp[\frac{i}{\hbar}S[h]] = K_h^{free}$ is the weight factor of the free action determined from the experiment and $L(h)$ denotes the classical length of path in the homotopy class h . In analogy to the regularization via Δt by Eq.(15), where $L(C)$ is given by the classical length of pieces of straight line, we define here $L(h)$ also by the classical length of a curve being an element of homotopy class h . It starts at x_{in} and arrives at x_{fi} . It goes by pieces of straight lines always passing in the middle of a pair of solenoids. Such regularization does not distinguish paths on a scale smaller than Δx . Thus the length $\langle \hat{L}(\Delta x) \rangle$ is obtained by taking K_h^{free} for homotopy class h from the experiment, construct $L(h)$ for homotopy class h from the array of flux tubes and compute the sum according to Eq.(16). This yields finally $\langle \hat{L}(\Delta x) \rangle$ in absence of the vector potential, i.e., corresponding to free propagation. Finally, in order to extract the Hausdorff dimension d_H , one has to measure the length $\langle \hat{L}(\Delta x) \rangle$ for many values of Δx , look for a power law behavior when $\Delta x \rightarrow 0$ and determine the critical exponent and thus d_H . As a consequence of the fact that this experiment is not sensitive to the zig-zagness parallel to the solenoids, we do not measure the length of the path but only its projection onto the plane perpendicular to the solenoids, i.e., in $D = 2$ dimensions. Nevertheless, the length as such is physically not so interesting (it depends on Δx anyway). The physically important quantity is the critical exponent (Hausdorff dimension) which corresponds to taking the limit $\Delta x \rightarrow 0$. But the latter should be the same in any number of space dimensions.

In summary, we have proposed a gedanken experiment how to measure the geometry of propagation of a massive particle in quantum mechanics. We have discussed the fundamental problem with experiments monitoring the path. We suggest to avoid the problem by doing an experiment sensitive to the topology of paths via a generalized Aharonov-Bohm experiment. This allows to determine homotopy classes and the Hausdorff dimension. We call it a gedanken experiment because we assume an idealized situation of infinitely thin flux tubes.

Acknowledgement

The author is grateful for discussions with Prof. Franson, Johns Hopkins University and

Prof. Hasselbach, Universität Tübingen. The author acknowledges support by NSERC Canada.

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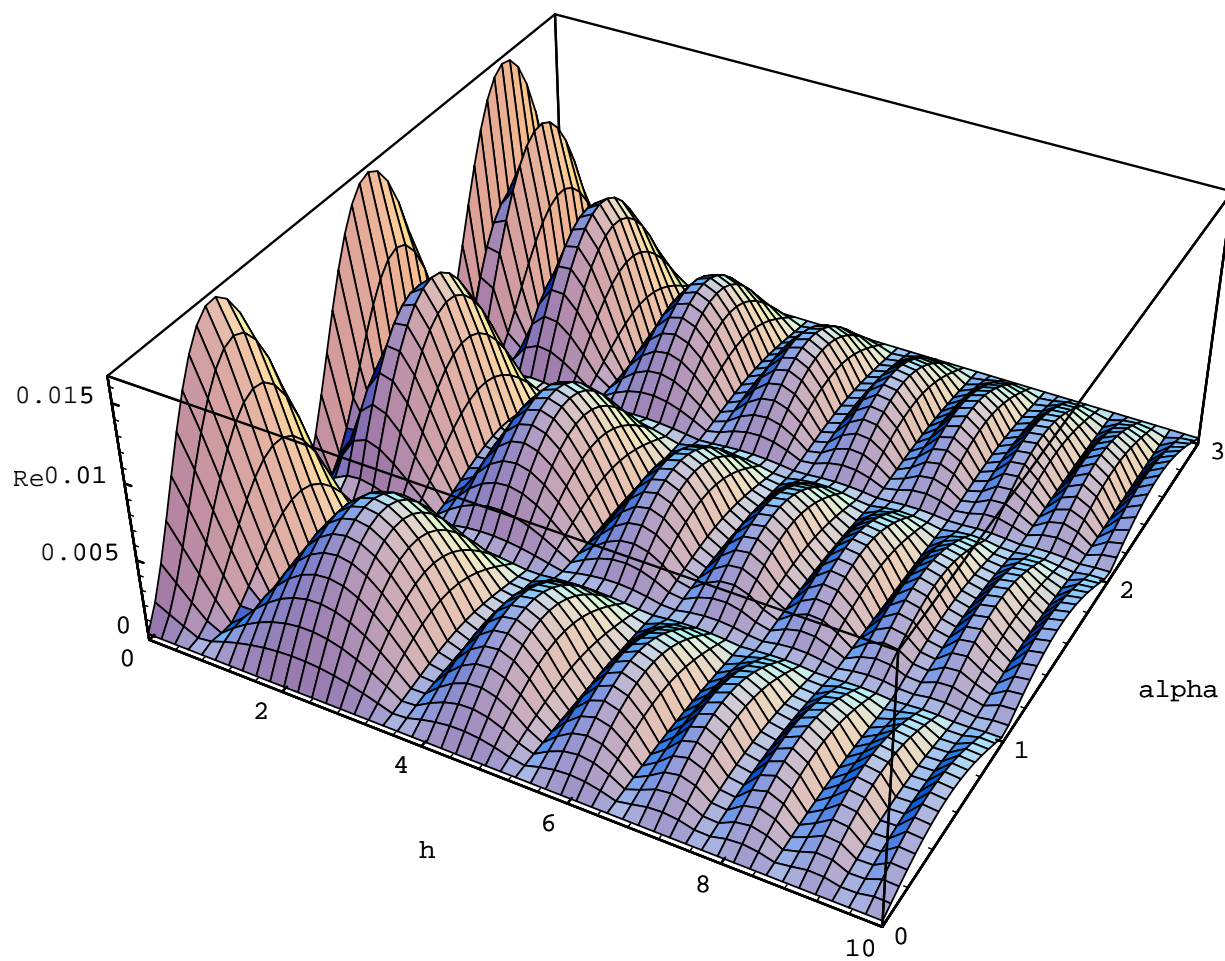
Figure Caption

Fig.1 Absolut value of real part of difference between Aharonov-Bohm propagator and semi-classical propagator. Dependence on distance h and on α . Cut-off $m_{max} = 50$.

Fig.2 Set-up of generalized Aharonov-Bohm experiment. There are N_S solenoids positioned in a regular array with distance Δx .

Fig.3 Example of two topologically equivalent paths.

Difference AB - semiclass. propagator



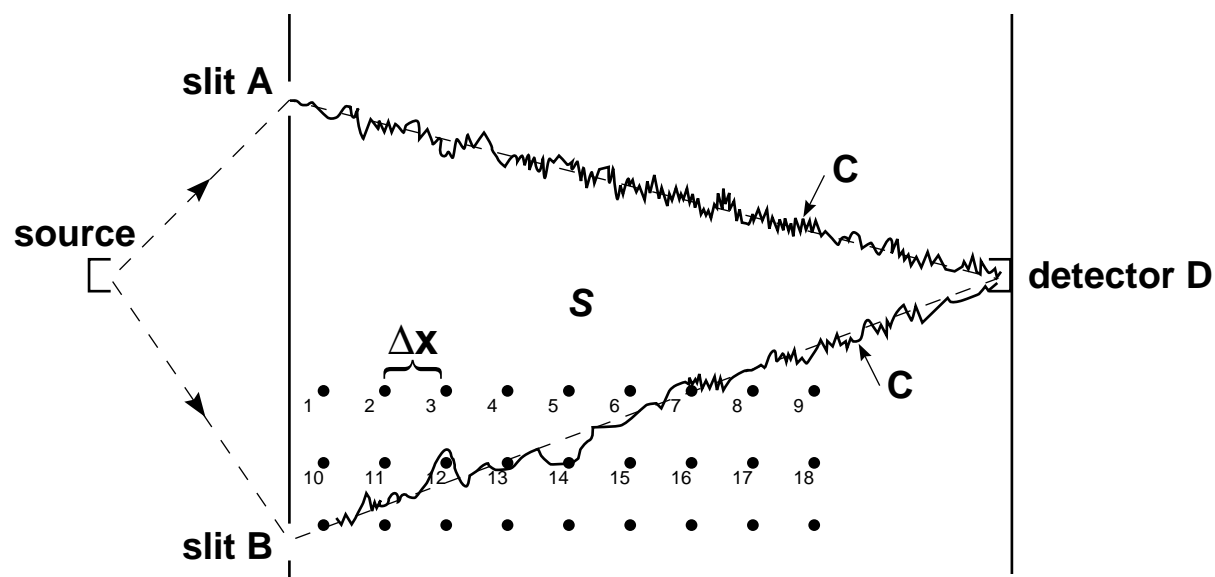


Fig. 14

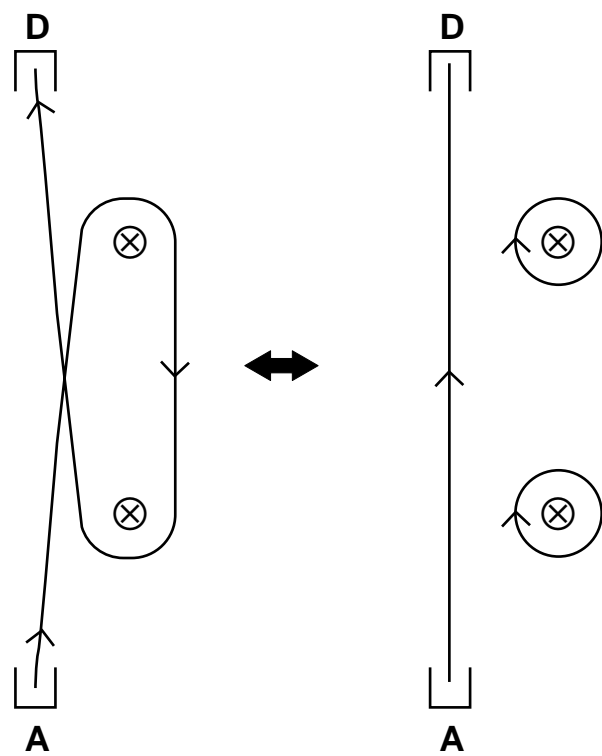


Fig. 15b